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Invariant solutions of hydrodynamic-type equations

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Abstract. Conditions are established under which a system of hydrodynamic-type equations with time-dependent coefficients allows an infinite-dimensional group of hydrodynamic-type symmetries. When this group exists it is used to linearize and solve the original system.

1. Introduction

The purpose of this paper is to apply Lie group theory to solve certain classes of quasilinear systems of first-order partial differential equations. The equations we study are a subclass of equations that have been called 'equations of the hydrodynamic type' [1–5].

The general form of such equations is

$$u_t^i = \sum_{j=1}^{p} \sum_{\alpha=1}^{n} v_\alpha^i(u^1, \ldots, u^n, t, x_1, \ldots, x_p) u_{x_j}^\alpha + B^i(u^1, \ldots, u^n, t, x_1, \ldots, x_p) \qquad 1 \leqslant i \leqslant n. \tag{1.1}$$

In the bulk of this paper we restrict ourselves to the case $p = 1,\ B^i = 0$, though some results for arbitrary p are presented. We also show that in certain cases inhomogeneous systems ($B \neq 0$), can be reduced to homogeneous ones. The number n of dependent functions $u^i(\vec{x}, t)$ is arbitrary in all cases and the coefficients v_α^i are explicitly space and time dependent.

An extensive literature exists on hydrodynamic-type equations [1–14]. Originally, Dubrovin and Novikov called 'systems of hydrodynamic type' quasilinear systems of first-order partial differential equations which possessed a Hamiltonian structure [1, 2]. Here we study a more general class of hydrodynamic-type equations which are quasilinear systems of first-order partial differential equations 'rich in symmetries' [3–5]. As a consequence of this last property they admit exact linearization and infinite series of their exact solutions can be constructed. Thus they have integrability properties which are as good as those of Hamiltonian equations. Hydrodynamic-type systems describe various physical phenomena: gas dynamics, hydrodynamics and magnetohydrodynamics [15–19], nonlinear elasticity, plasticity and phase-transition models [20, 21], chromatography and electrophoresis equations from physical chemistry and biology [22, 23]. Another class of applications is obtained by a representation of physically interesting higher-order equations as integrability conditions of hydrodynamic-type systems. These include, in particular, the Euler and Poisson equations of nonlinear acoustics [24], the Born–Infeld equation of nonlinear electrodynamics [25] and systems of relativistic-string equations [26]. Modern applications of hydrodynamic-type systems arise in the theory of averaging nonlinear soliton equations [2, 4, 14].

In section 2 we concentrate on hydrodynamic-type equations with one space variable and an explicit time dependence in the coefficients. We establish the conditions under which these equations have an infinite set of 'hydrodynamic symmetries'. These are represented by flows, commuting with those of the equations under study. The flows depend linearly on the space derivatives u^i_x, but they do not, in general, correspond to Lie point symmetries. Once the existence conditions are satisfied, we obtain determining equations for the symmetries. To obtain these symmetries explicitly we must solve a system of linear first-order partial differential equations. Formulae for the corresponding invariant solutions amount to a linearizing transformation. The solution of the nonlinear hydrodynamic-type equations is reduced to the problem of solving a system of linear equations. The explicit linearizing transformation is given in section 3. The method presented in sections 2 and 3 is a generalization of the method suggested by Tsarev [3, 4] to the case of time-dependent coefficients.

Section 4 is devoted to the construction of first- and second-order recursion operators. These operators transform solutions of the linear system amongst each other. They are used in section 5 to generate an infinite series of solutions of the linear system and hence of the nonlinear hydrodynamic equations themselves. Some preliminary results in this direction were obtained earlier by one of the authors [7–11].

A specific example when the entire procedure works explicitly is treated in section 6. In section 7 we show how an inhomogeneous system can be reduced to a homogeneous one. Finally, in section 8 we discuss hydrodynamic-type equations with n space variables and relate Lie symmetry methods to the method of Riemann waves [15–17, 19, 27].

2. Hydrodynamic equations with one space variable

2.1. Hydrodynamic flows and a generalized concept of symmetry

In this section we discuss the n-component homogeneous quasilinear strictly hyperbolic system of first-order partial differential equations with one time (t) and one space (x) variable and n fields $u = (u^1, \dots, u^n)$. We assume that the system is diagonalizable and may depend on t explicitly. We consider it in the diagonal form

$$u^i_t = v_i(u, t)u^i_x \qquad i = 1, 2, \dots, n \qquad n \geqslant 3 \tag{2.1}$$

Here and subsequently the subscripts t, x and u^i denote partial derivatives with respect to these indices. All summations will be shown explicitly. In particular, there is no summation over i in equation (2.1). The condition for the system (2.1) to be strictly hyperbolic means the non-degeneracy of its spectrum $v_i \neq v_j$ for $i \neq j$. We search for hydrodynamic flows commuting with the flow determined by the original system (2.1). This means that the flows we are searching for are also determined by hydrodynamic-type equations,

$$u^i_\tau = \sum_{j=1}^{n} A^i_j(u, t, x)u^j_x \qquad i = 1, 2, \dots, n \tag{2.2}$$

where we assume that the unknowns u^i depend upon an additional independent variable τ related to each flow (2.2). Here only the unknowns u^i are supposed to be transformed by the flow (2.2), while the independent variables t, x are invariant, satisfying the equalities $t_\tau = 0, x_\tau = 0$. We say that the flows (2.1) and (2.2) commute if the compatibility conditions $u^i_{t\tau} = u^i_{\tau t}$ are satisfied identically for $i = 1, \ldots, n$ as a consequence of equations (2.1) and (2.2). We can regard such flows as generalized symmetries and call them *hydrodynamic symmetries* in the sense that the flows (2.2) leave the original system (2.1) invariant (since the flow (2.1) does not depend on τ). Such an interpretation is also possible for usual Lie symmetries such as point symmetries or higher-order symmetries. In these cases the flows (2.2) are Lie equations of the corresponding one-parameter symmetry groups [28, 29]. We mention that point symmetries are generated by the vector fields of the form

$$\hat{X} = \xi(x, t, u)\partial_x + \eta(x, t, u)\partial_t + \sum_{j=1}^{n} \phi_j(x, t, u)\partial_{u^j} \tag{2.3}$$

and for these the flow equations (2.2) are

$$u^i_\tau = -\xi(x, t, u)u^i_x - \eta(x, t, u)u^i_t + \phi_i(x, t, u). \tag{2.4}$$

Thus, hydrodynamic symmetries are point symmetries only if we have $A^i_j(x, t, u) \equiv A(x, t, u)$ for all i and j. Hydrodynamic-type systems of the general form (2.1) do not admit a symmetry group of point transformations apart from the obvious translational symmetry in x. However, under certain conditions, which are obtained below, equations (2.1) do admit an infinite continuous set of commuting flows which cannot be interpreted as Lie equations for their local symmetries. Nevertheless, one can understand them as generalized symmetries of this system because the flows leave the original system (2.1) invariant. Furthermore, we can apply these symmetries to obtain families of new solutions from known ones. Indeed, let $\varphi(x, t)$ be a solution of the system (2.1) and let us solve the Cauchy problem for the system (2.2) with the initial condition $u^i(x, t, \tau)|_{\tau=0} = \varphi^i(x, t)$. The solution $u(x, t, \tau)$ of this Cauchy problem presents a one-parameter family of solutions of the original system (2.1) which are generated by the flow (2.2) from the known solution $\varphi(x, t)$. This manner of generating new solutions is similar to the standard application of Lie symmetries of differential equations. Another important application of Lie symmetries is symmetry reduction, i.e. the search for invariant solutions. We can also use generalized symmetries generated by the hydrodynamic flows (2.2) for this purpose by imposing the constraint $u^i_\tau = 0$ and simultaneously solving equations (2.1) and (2.2).

2.2. Conditions for the existence of an infinite-dimensional algebra of flows

Let us search for hydrodynamic flows of the form (2.2) leaving x and t invariant, i.e. $x_\tau = 0, t_\tau = 0$, and commuting with the diagonal explicitly t-dependent hydrodynamic-type system (2.1), subject to the non-degeneracy condition $v_i \neq v_j$ for $i \neq j$. We calculate the left-hand side of the equality $u^i_{t\tau} - u^i_{\tau t} = 0$ by differentiating equations (2.1) and (2.2) with respect to τ and t, respectively, and then using the original equations to exclude derivatives in t and τ. We end up with equations for $A^i_j(u, t, x)$ with arbitrary fixed $v_i(u, t)$ involving only derivatives in x of the unknowns u^i up to the second order $\{u^i_x, u^i_{xx}\}$. The functions $v_i(u, t)$ and $A^i_j(u, t, x)$ do not depend upon the derivatives u^i_x, u^i_{xx} and hence the coefficients of the terms with different dependences on the derivatives must be zero. This results in a splitting of the compatibility conditions $u^i_{t\tau} = u^i_{\tau t}$ for each value of i into a set of equations for A^i_j which do not contain any derivatives of u^i in x. The coefficients multiplying terms involving

u^i_{xx} must vanish. This implies that the matrix A^i_j must be diagonal $A^i_j = \delta^i_j A_i$, where δ^i_j is the Kronecker symbol. Hence the hydrodynamic flows (2.2) commuting with the original diagonal flow (2.1) must be diagonal

$$u^i_\tau = A_i(u,t,x)u^i_x \qquad i = 1,2,\ldots,n. \tag{2.5}$$

Putting coefficients of the terms bilinear in u^i_x equal to zero, we obtain the equalities

$$v_{i,u^j}(A_j - A_i) = A_{i,u^j}(v_j - v_i) \qquad j \neq i. \tag{2.6}$$

This leads us in a natural way to consider the quantities

$$c^i_{ij}(u,t) = \frac{v_{i,u^j}(u,t)}{v_j(u,t) - v_i(u,t)} \qquad i \neq j. \tag{2.7}$$

Their geometrical meaning is that they constitute a family of connection coefficients of a fibering associated with the system (2.1) which depends upon the parameter t. In the case when the system (2.1) does not depend explicitly on t, the connection coefficients $\Gamma^i_{ij}(u)$ (independent of t) were introduced by Tsarev (see [4] and references therein). In our t-dependent case these quantities also arise as initial values for $t = 0$ of our family of connection coefficients

$$\Gamma^i_{ij}(u) = c^i_{ij}(u,0) \qquad j \neq i. \tag{2.8}$$

With this notation equations (2.6) take the form

$$A_{i,u^j}(u,t,x) = c^i_{ij}(u,t)(A_j - A_i) \qquad j \neq i. \tag{2.9}$$

A further set of equations for A^i is obtained when we set equal to zero the coefficients of the terms linear in u^i_x

$$A_{i,t}(u,t,x) = v_i(u,t)A_{i,x}(u,t,x) \qquad i = 1,2,\ldots,n. \tag{2.10}$$

A complete analysis of compatibility conditions of the two systems (2.9) and (2.10) is too lengthy to be presented here. In the remaining part of this section we only present the final results of this analysis in the generic case. By this we mean that all necessary restrictions were imposed on the coefficients of the original system (2.1) in order to keep maximal freedom for the coefficients A_i of the flows (2.5).

The main result is stated in the following theorem (preliminary results were obtained in the preprint [7]).

Theorem 2.1. *The diagonal n-component hydrodynamic-type system (2.1) with an explicit t dependence admits an infinite set of hydrodynamic symmetries of the form (2.2) with a functional arbitrariness if its coefficients satisfy the condition (for $n \geqslant 3$)*

$$\left[\frac{v_{i,u^j}(u,t)}{v_j(u,t) - v_i(u,t)}\right]_{u^k} = \left[\frac{v_{i,u^k}(u,t)}{v_k(u,t) - v_i(u,t)}\right]_{u^j} \iff c^i_{ij,u^k}(u,t) = c^i_{ik,u^j}(u,t)$$
$$i \neq j \neq k \neq i \tag{2.11}$$

and the condition

$$\left[\frac{v_{i,u^j}(u,t)}{v_j(u,t) - v_i(u,t)}\right]_t = -\beta v_{i,u^j}(u,t) \iff c^i_{ij,t}(u,t) = -\beta\left(v_j - v_i\right)c^i_{ij}(u,t) \qquad i \neq j \tag{2.12}$$

where β is an arbitrary (real) constant. These symmetries are generated by the hydrodynamic flows (2.5) with coefficients A_i defined by the formulae

$$A_i(u, t, x) = w_i(u) \exp\left\{\beta\left[x + \int_0^t v_i(u, t)\,\mathrm{d}t\right]\right\} + C \tag{2.13}$$

(for $\beta \neq 0$),

$$A_i(u, t, x) = w_i(u) + C\left[x + \int_0^t v_i(u, t)\,\mathrm{d}t\right] \tag{2.14}$$

(for $\beta = 0$), where C is an arbitrary constant. The integrations over t are performed for a constant value of u. Here the set of functions $\{w_i(u)\}$ constitutes an arbitrary smooth solution of the linear system

$$w_{i,u^j}(u) = \Gamma^i_{ij}(u)\left(w_j - w_i\right) \qquad i \neq j \tag{2.15}$$

with the coefficients Γ^i_{ij} defined by the formulae (2.8) and (2.7).

The solution manifold of the system (2.15) depends upon n arbitrary functions $c_i(u^i)$ of one variable which locally parametrize the set of hydrodynamic flows.

The following remarks are in order.

(a) The condition (2.11) entails the local existence of a vector potential $\phi_i(u, t)$ satisfying the equalities

$$c^i_{ij}(u, t) = \phi_{i,u^j}(u, t) \qquad i, j = 1, 2, \ldots, n. \tag{2.16}$$

(b) The condition (2.11) in view of the definition (2.7) for $c^i_{ij}(u, t)$ implies (for $n \geqslant 3$) the following equalities:

$$c^i_{ij,u^k}(u, t) = c^i_{ik}c^k_{kj} + c^i_{ij}\left(c^j_{jk} - c^i_{ik}\right) \qquad i \neq j \neq k \neq i \tag{2.17}$$

that also imply equations (2.11).

(c) For $t = 0$ equations (2.17) reduce to

$$\Gamma^i_{ij,u^k}(u) = \Gamma^i_{ik}\Gamma^k_{kj} + \Gamma^i_{ij}\left(\Gamma^j_{jk} - \Gamma^i_{ik}\right) \qquad i \neq j \neq k \neq i. \tag{2.18}$$

(d) The condition (2.12) determines the explicit t dependence of the coefficients $c^i_{ij}(u, t)$ ($i \neq j$):

$$c^i_{ij}(u, t) = \Gamma^i_{ij}(u) \exp\left\{-\beta \int_0^t \left[v_j(u, t) - v_i(u, t)\right]\mathrm{d}t\right\}.$$

(e) We can use the freedom in the definitions (2.7) and (2.16) of the functions c^i_{ij} and ϕ_i to transform the condition (2.12) to a simpler form:

$$\phi_{i,t}(u, t) = -\beta v_i(u, t) \qquad i = 1, 2, \ldots, n. \tag{2.19}$$

(f) For two-component systems ($n = 2$) the first condition (2.11) of theorem 2.1 (and also equations (2.17) and (2.18)) must be omitted and hence only the second condition (2.12) is taken into account.

We mention that the results obtained, as well as all the results obtained below in sections 3–6, can be easily transferred from explicitly t-dependent systems (2.1) to the explicitly x-dependent systems

$$u^i_t = v_i(u, x)u^i_x \qquad i = 1, 2, \ldots, n. \tag{2.20}$$

This system can be transformed to the form (2.1) by a simple change of notation

$$t \mapsto x \qquad x \mapsto t \qquad v_i(u, x) \mapsto \frac{1}{v_i(u, t)}. \tag{2.21}$$

3. Invariant solutions and linearizing transformations

The conditions determining invariant solutions of explicitly t-dependent systems (2.1), subject to the 'non-singularity' constraint: $u^i_x \neq 0$, are obtained by putting $u^i_\tau = 0$ in equations (2.5) determining the commuting hydrodynamic flows. They have the form $A_i(u, t, x) = 0$ with the functions A_i defined by the formulae (2.13) or (2.14). Hence we obtain (for $C = 1$ and $i = 1, 2, \ldots, n$):

$$\beta \neq 0: \qquad w_i(u) + \exp\left\{-\beta\left[x + \int_0^t v_i(u, t)\,\mathrm{d}t\right]\right\} = 0 \tag{3.1}$$

$$\beta = 0: \qquad w_i(u) + x + \int_0^t v_i(u, t)\,\mathrm{d}t = 0. \tag{3.2}$$

Here the set of functions $w_i(u)$ is an arbitrary smooth solution of the linear system (2.15). Thus the equalities (3.1) and (3.2) determine linearizing transformations for the t-dependent system (2.1): solving this system reduces to solving the linear system (2.15). These equalities determine the whole non-singular solution manifold $u^i = u^i(x, t)$ of the original nonlinear system, i.e. its 'general solution', if the conditions of the implicit-function theorem are satisfied, i.e. the Jacobian determinant $\det(w_{i,u^j}(u))$ is non-zero. Then the equalities (3.1) and (3.2) can be solved with respect to u in the general form

$$u^i = f_i(r^1, r^2, \ldots, r^n) \qquad i = 1, 2, \ldots, n \tag{3.3}$$

which determines the implicit solution in the form of Riemann waves with Riemann invariants r^i

$$r^i = x + \int_0^t v_i(u, t)\,\mathrm{d}t \qquad i = 1, \ldots, n. \tag{3.4}$$

A number of attempts to generalize the Riemann invariant method and its various applications can be found in the recent literature on the subject (see, e.g., [27, 30–33] and references therein).

Thus for the considered hydrodynamic-type systems the general solution consists of solutions invariant with respect to the hydrodynamic flows (2.5). Hence the existence of linearizing transformations has a group-theoretical reason: the equations admit an infinite continuous set of hydrodynamic symmetries generated by the flows (2.5) and the degree of generality of these symmetries is the same as that of the general solution of equation (2.1).

In the remaining part of this section we prove that the invariant solutions actually exist, i.e. that the equalities (3.1) and (3.2) are compatible with the original hydrodynamic-type system. We shall do this explicitly only for equation (3.2), i.e. in the case $\beta = 0$. In the case $\beta \neq 0$ the proof is similar.

First, we take the difference of equation (3.2) and the same equation with i replaced by $j \neq i$

$$w_j - w_i = \int_0^t [v_i(u, t) - v_j(u, t)]\,\mathrm{d}t. \tag{3.5}$$

Then we take the derivatives of equation (3.2) with respect to t and x (taking the dependence of u^i on t and x into account)

$$\begin{aligned} &\sum_{j=1}^{n} w_{i,u^j} u_t^j + v_i(u,t) + \sum_{j=1}^{n} u_t^j \int_0^t v_{i,u^j}(u,t)\,\mathrm{d}t = 0 \\ &\sum_{j=1}^{n} w_{i,u^j} u_x^j + 1 + \sum_{j=1}^{n} u_x^j \int_0^t v_{i,u^j}(u,t)\,\mathrm{d}t = 0. \end{aligned} \tag{3.6}$$

Next we use the linear equations (2.15) to exclude the derivatives of w_i in equation (3.6). Similar equations for v_i follow from the definitions (2.7) and (2.8) in the case of $\beta = 0$

$$v_{i,u^j}(u,t) = \Gamma^i_{ij}(u)(v_j - v_i) \qquad j \neq i. \tag{3.7}$$

Equation (3.5) is used to exclude the differences $w_j - w_i$. The result is that all non-diagonal terms cancel and equations (3.6) decouple and are easily solved with respect to the derivatives of u^i

$$u_t^i = -\frac{v_i(u,t)}{w_{i,u^i}(u) + \int_0^t v_{i,u^i}(u,t)\,\mathrm{d}t} \qquad u_x^i = -\frac{1}{w_{i,u^i}(u) + \int_0^t v_{i,u^i}(u,t)\,\mathrm{d}t}. \tag{3.8}$$

Substituting these expressions into equations (2.1) we see that they are identically satisfied.

Equations (3.8) are meaningful if the denominators on the right-hand side are non-zero. The conditions

$$w_{i,u^i}(u) + \int_0^t v_{i,u^i}(u,t)\,\mathrm{d}t \neq 0 \qquad i = 1, \ldots, n \tag{3.9}$$

coincide with the conditions of the implicit function theorem, applied to equation (3.2) (i.e. the condition for $u^i(x,t)$ to be determined implicitly by equation (3.2)).

The results of this section can be summed up as follows. If the conditions of theorem 2.1 are satisfied, then we can obtain an infinite set of hydrodynamic symmetries by solving the linear system (2.15) for $w_i(u^1, \ldots, u^n)$. Once the solutions $w_1, \ldots, w_n$ are known, then the relations (3.1) for $\beta \neq 0$, or (3.2) for $\beta = 0$, constitute an implicit solution of the hydrodynamic-type equations (2.1).

4. Recursions of symmetries and exact solutions

We define a *recursion operator* as an operator which maps any symmetry of a given system into a symmetry of the same system. In our case there exists a restriction of this operator onto the linear subspace of hydrodynamic symmetries generated by equations (2.5) with coefficients which depend upon the choice of solution of the linear system (2.15) [7–11]. The action of this operator is reduced to a transformation of the original solution $\{w_i(u)\}$ of equations (2.15) into some new solution $\{\widehat{w}_i(u)\}$. This recursion of the solutions, applied to the linearizing transformations, gives rise to a recursion of solutions of the given nonlinear system (2.1). We shall call the *order of recursion* the maximal order of the derivatives of $w_i(u)$ involved in the recursion formula.

4.1. First-order recursion

Let us define the functions ($i = 1, 2, \ldots, n$):

$$S_i(u) = \sum_{k=1}^{n} \Gamma^i_{ik}(u) c_k(u^k) + d_i(u^i). \tag{4.1}$$

They depend upon $2n$ functions $c_i(u^i)$, $d_i(u^i)$ of one variable each.

Theorem 4.1. *Let the system (2.1) satisfy the conditions of theorem 2.1. Then there exists a first-order recursion of its hydrodynamic symmetries and hence a recursion of solutions of the linear system (2.15), iff there exist* $2n$ *functions* $c_i(u^i)$, $d_i(u^i)$ *of one variable which satisfy the conditions*

$$S_{i,u^j}(u) = \Gamma^i_{ij}(u)(S_j - S_i) \qquad i \neq j \tag{4.2}$$

with the functions $S_i(u)$ *defined by equations (4.1). This recursion is determined by the formula:*

$$\widehat{w}_i(u) = c_i(u^i)w_{i,u^i}(u) + d_i(u^i)w_i(u) + \sum_{k=1}^{n} \Gamma^i_{ik}(u)c_k(u^k)w_k(u). \tag{4.3}$$

For hydrodynamic-type systems independent of t and x this result was obtained by Teshukov [34].

Corollary 4.1. *For any solution* $\{w_i(u)\}$ *of the linear system (2.15) with coefficients* $\Gamma^i_{ij}(u)$ *satisfying equalities of the form* (2.11)

$$\Gamma^i_{ij,u^k}(u) = \Gamma^i_{ik,u^j}(u) \qquad i \neq j \neq k \neq i \tag{4.4}$$

the functions $\widehat{w}_i(u)$ *of* equation (4.3) *also form a solution of this system iff the conditions (4.2) are satisfied.*

The following remark is in order. The linearizing transformations (3.1) and (3.2) reduce the search for any 'non-singular' solution of the nonlinear system (2.1) to a search for a solution of the linear system (2.15). However, the integration of linear equations (2.15) with variable coefficients can also pose a problem. The existence of a recursion for symmetries allows one to obtain new solutions of this linear system from old ones and hence to obtain new solutions of nonlinear hydrodynamics-type systems.

4.2. Second-order recursion

Let us define the 'connection potential' $V(u)$ by a completely integrable (in the Frobenius sense) system:

$$V_{u^iu^j}(u) = \Gamma^i_{ij}\Gamma^j_{ji} \qquad i \neq j. \tag{4.5}$$

Its solution $V(u)$ depends upon n arbitrary functions of one variable. The integrability conditions for the system (4.5) are satisfied by virtue of the semi-Hamiltonian property (4.4).

Let us define the functions

$$b_{ik}(u) = f_k(u^k)\left[\Gamma^i_{ik}\left(2\Gamma^k_{kk} - \Gamma^i_{ik}\right) - \Gamma^i_{ik,u^k}\right] + \left[c_k(u^k) - f'_k(u^k)\right]\Gamma^i_{ik} \qquad i \neq k \tag{4.6}$$

$$b_{ii}(u) = f_i(u^i)\left[\Gamma^i_{ii,u^i} + \left(\Gamma^i_{ii}\right)^2 - 2V_{u^iu^i}(u)\right] - f'_i(u^i)V_{u^i}(u) + c_i(u^i)\Gamma^i_{ii} + d_i(u^i) \tag{4.7}$$

$$B_i(u) = \sum_{k=1}^{n} b_{ik}(u) \tag{4.8}$$

which depend upon $3n$ functions $f_i(u^i)$, $c_i(u^i)$, $d_i(u^i)$ of one variable each.

Theorem 4.2. *Let the system (2.1) satisfy the conditions of theorem 2.1. Then there exists a second-order recursion of the hydrodynamic symmetries (of solutions of the linear system (2.15)) iff there exist* $3n$ *functions* $f_i(u^i)$, $c_i(u^i)$, $d_i(u^i)$ *of one variable which satisfy the conditions*

$$B_{i,u^j}(u) = \Gamma^i_{ij}(u)(B_j - B_i) \qquad i \neq j \tag{4.9}$$

with the functions $B_i(u)$ defined by equations (4.6)–(4.8). This recursion is determined by the formula ($i = 1, 2, \ldots, n$)*:*

$$\widehat{w}_i(u) = f_i(u^i)w_{i,u^iu^i} + \left[2f_i(u^i)\Gamma^i_{ii} + c_i(u^i)\right] w_{i,u^i} + \sum_{k\neq i} f_k(u^k)\Gamma^i_{ik}w_{k,u^k} + \sum_{k=1}^{n} b_{ik}(u)w_k \tag{4.10}$$

where the functions $b_{ik}(u)$ are defined by the formulae (4.6) and (4.7).

Hence for any solution $\{w_i(u)\}$ of the linear system (2.15) the formula (4.10) also gives a solution $\{\widehat{w}_i(u)\}$ of this system if and only if the conditions (4.9) are satisfied.

Theorem 4.3. *If a first-order recursion operator exists then there also exists a second-order recursion operator equal to the squared first-order recursion operator. The inverse is not true, i.e. existence conditions (4.2) for first-order recursions do not follow from the existence conditions (4.9) for second-order recursion operators.*

This means that the existence conditions for second-order recursions are less restrictive than those for first-order ones.

The second-order recursion operator for systems of the form (2.1), but independent of t (and x), and theorem 4.3 were obtained earlier [8, 10].

5. Generation of an infinite series of exact solutions

To obtain explicit formulae for invariant solutions of the nonlinear system (2.1) one must search for solutions of the linear system (2.15) and substitute these solutions for a set of functions $\{w_i(u)\}$ in the linearizing transformations (3.1) or (3.2).

The linear system (2.15) has two trivial solutions:

$$w_i = 1 \qquad w_i = v_i^0(u) \qquad i = 1, 2, \ldots, n \tag{5.1}$$

where $v_i^0(u) = v_i(u, 0)$. They serve as starting elements for generating infinite series of non-trivial solutions by means of a repeated application of the recursion operators.

Specifically, let us assume that a first-order recursion (4.3) exists for solutions of equations (2.15), i.e. the conditions (4.2) are satisfied. We can write it in the operator form:

$$\widehat{w}_i(u) = (R_1[w])_i = \sum_{k=1}^{n}(R_1)_{ik}[w_k] \tag{5.2}$$

with the first-order operator R_1 defined as

$$(R_1)_{ik} = \delta_{ik}\left[c_i(u^i)\frac{\partial}{\partial u^i} + d_i(u^i)\right] + \Gamma^i_{ik}(u)c_k(u^k). \tag{5.3}$$

The two trivial solutions (5.1) are mapped by the operator R_1 into non-trivial solutions of the system (2.15):

$$\widehat{w}_i(u) = (R_1[1])_i = \sum_{k=1}^{n}\Gamma^i_{ik}(u)c_k(u^k) + d_i(u^i) \equiv S_i(u) \tag{5.4}$$

$$\widehat{w}_i(u) = \left(R_1[v^0]\right)_i = c_i(u^i)v^0_{i,u^i}(u) + d_i(u^i)v_i^0(u) + \sum_{k=1}^{n}\Gamma^i_{ik}(u)c_k(u^k)v_k^0(u). \tag{5.5}$$

Substituting these expressions for $w_i(u)$ into the linearizing transformations (3.1) or (3.2) we obtain explicit formulae for the non-trivial solutions of the system (2.1). In particular, if the coefficients v_i in equation (2.1) depend on u alone, we obtain the solutions ($i = 1, 2, \ldots, n$):

$$\sum_{k=1}^{n} \Gamma^i_{ik}(u) c_k(u^k) + d_i(u^i) = t v_i(u) + x \tag{5.6}$$

$$c_i(u^i) v_{i,u^i}(u) + d_i(u^i) v_i(u) + \sum_{k=1}^{n} \Gamma^i_{ik}(u) c_k(u^k) v_k(u) = t v_i(u) + x. \tag{5.7}$$

These equalities determine exact solutions of such a system as implicit functions $u^i = u^i(x, t)$.

A repeated action of the operator R_1 upon the trivial solutions (5.1)

$$w_i = \left(R_1^N[1]\right)_i \qquad w_i = \left(R_1^N[v^0]\right)_i$$

and the use of these expressions for w_i in linearizing transformations generates explicit formulae for two infinite series of exact invariant solutions of t- and x-dependent systems. In particular, for t- and x-independent systems they have the form ($N = 1, 2, \ldots$):

$$\left(R_1^N[1]\right)_i = t v_i(u) + x \qquad \left(R_1^N[v]\right)_i = t v_i(u) + x. \tag{5.8}$$

Now assume that the less restrictive conditions (4.9) are met for the existence of a second-order recursion. Then the recursion (4.10) is valid for solutions of the linear system (2.15). We write it again in the operator form

$$\widehat{w}_i(u) = (R_2[w])_i = \sum_{k=1}^{n} (R_2)_{ik}[w_k] \tag{5.9}$$

with the second-order operator

$$(R_2)_{ik} = \delta_{ik} \left\{ f_i(u^i) \frac{\partial^2}{(\partial u^i)^2} + \left[f_i(u^i) \Gamma^i_{ii}(u) + c_i(u^i) \right] \frac{\partial}{\partial u^i} \right\} + f_k(u^k) \Gamma^i_{ik}(u) \frac{\partial}{\partial u^k} + b_{ik}(u) \tag{5.10}$$

and the functions $b_{ik}(u)$ defined by equations (4.6) and (4.7). The trivial solutions (5.1) are mapped by the operator R_2 to the non-trivial solutions of the system (2.1) ($i = 1, 2, \ldots, n$):

$$\begin{aligned} \widehat{w}_i &= (R_2[1])_i \equiv \sum_{k=1}^{n} b_{ik}(u) \equiv B_i(u) \\ \widehat{w}_i &= (R_2[v^0])_i \equiv f_i(u^i) v^0_{i,u^i u^i}(u) + \left[2 f_i(u^i) \Gamma^i_{ii}(u) + c_i(u^i) \right] v^0_{i,u^i}(u) \\ &\quad + \sum_{k \neq i} f_k(u^k) \Gamma^i_{ik}(u) v^0_{k,u^k}(u) + \sum_{k=1}^{n} b_{ik}(u) v^0_k(u). \end{aligned} \tag{5.11}$$

Substituting these expressions for $w_i(u)$ into the linearizing transformations (3.1) and (3.2) we obtain explicit formulae for the non-trivial solutions of the system (2.1).

In particular, for t- and x-independent systems we obtain the solutions ($i = 1, 2, \ldots, n$):

$$\begin{aligned} &\sum_{k=1}^{n} b_{ik}(u) \equiv B_i(u) = t v_i(u) + x \\ &f_i(u^i) v_{i,u^i u^i}(u) + \left[2 f_i(u^i) \Gamma^i_{ii}(u) + c_i(u^i) \right] v_{i,u^i}(u) \\ &\qquad + \sum_{k \neq i} f_k(u^k) \Gamma^i_{ik}(u) v_{k,u^k}(u) + \sum_{k=1}^{n} b_{ik}(u) v_k(u) = t v_i(u) + x. \end{aligned} \tag{5.12}$$

The action of powers of the operator R_2 upon the trivial solutions (5.1):

$$w_i = \left(R_2^N[1]\right)_i \qquad w_i = \left(R_2^N[v^0]\right)_i$$

and the use of these expressions for w_i in linearizing transformations generates the explicit formulae for two infinite series of exact invariant solutions of t- and x-dependent systems. In particular, for t- and x-independent systems they have the form ($N = 1, 2, \ldots$)

$$\left(R_2^N[1]\right)_i = tv_i(u) + x \qquad \left(R_2^N[v]\right)_i = tv_i(u) + x. \tag{5.13}$$

We can also use linear combinations of solutions of the two series for the functions $w_i(u)$ in linearizing transformations. For example, for t- and x-independent systems we have (with any integer N, M):

$$C_1\left(R_{1,2}^N[1]\right)_i + C_2\left(R_{1,2}[v]\right)_i = tv_i(u) + x \tag{5.14}$$

where R_1 or R_2 must be chosen and C_1 and C_2 are arbitrary constants.

We mention that the operators R_1 and R_2 coincide with the first- and second-order symmetry operators for the linear system (2.15) essential for the separation of variables in these equations. A solution of the system (2.15) by means of a separation of variables would mean completely solving the original nonlinear system. Hence the combined utilization of linearizing transformations and recursions may be considered as a transfer of the method of separation of variables to nonlinear systems.

6. Example of solving a system admitting an infinite set of hydrodynamic symmetries

6.1. Derivation of a class of systems with an infinite set of commuting hydrodynamic flows

In this section we show how the conditions of theorem 2.1 in the case of $\beta = 0$, combined with a natural ansatz on the form of the coefficients $v_i(u,t)$ can determine a system of the form (2.1) admitting an infinite set of commuting hydrodynamic flows (2.5). We obtain its general solution using hydrodynamic symmetries.

In the case of $\beta = 0$ the second set (2.12) of conditions of theorem 2.1 takes the form

$$c^i_{ij,t}(u,t) = 0 \qquad i \neq j \tag{6.1}$$

which in view of the relation (2.8) means that the connection coefficients

$$c^i_{ij} = \Gamma^i_{ij}(u) \qquad i \neq j \tag{6.2}$$

do not depend explicitly on t.

The first set of conditions of theorem 2.1 is satisfied by setting

$$\Gamma^i_{ij}(u) = \psi^i_{u^j}(u) \qquad i \neq j \tag{6.3}$$

where we introduced the connection potentials $\psi^i(u)$. Using the definition (2.7) of $c^i_{ij}(u)$ and the relations (6.2) and (6.3) we obtain

$$v_{i,u^j}(u,t) = \psi^i_{u^j}(u)(v_j - v_i) \iff \frac{v_{i,u^j}(u,t)}{v_j - v_i} = \psi^i_{u^j}(u) \qquad j \neq i. \tag{6.4}$$

The fact that the ratios (6.4) of t-dependent quantities do not depend explicitly on t implies that the simplest natural way for the numerator and the denominator to depend on t is to have the entire explicit t dependence in a common t-dependent factor

$$\begin{aligned} v_{i,u^j}(u,t) &= \alpha_{ji}(t)\psi^i_{u^j}(u)\omega^j_i(u) \\ v_j(u,t) - v_i(u,t) &= \alpha_{ji}(t)\omega^j_i(u) \end{aligned} \qquad j \neq i. \tag{6.5}$$

The condition $v_i \neq v_j$ for $i \neq j$ for the system (2.1) to be strictly hyperbolic implies the constraint

$$\alpha_{ji}(t)\omega_i^j(u) \neq 0 \qquad \text{for} \quad j \neq i. \tag{6.6}$$

The obvious relation $(v_k - v_j) + (v_j - v_i) = v_k - v_i$ yields the equalities

$$\alpha_{kj}(t)\omega_j^k(u) + \alpha_{ji}(t)\omega_i^j(u) = \alpha_{ki}(t)\omega_i^k(u) \qquad i \neq j \neq k. \tag{6.7}$$

There are only two simple ways to satisfy these equations.

(a) To have a common factor depending on u on the left-hand side of equations (6.7)

$$\omega_j^k(u) = \omega_i^j(u) = \omega_i^k(u) \equiv \omega(u) \qquad i \neq j \neq k. \tag{6.8}$$

(b) To have a common factor depending on t on the left-hand side of equations (6.7)

$$\alpha_{kj}(t) = \alpha_{ji}(t) = \alpha_{ki}(t) \equiv \alpha(t) \qquad i \neq j \neq k. \tag{6.9}$$

We drop here admissible but inessential constant factors which could be absorbed by a suitable redefinition of α_{ji} or ω_i^j.

The complete analysis of the case (b) leads to the obvious implementation of t dependence of v_i,

$$v_i(u,t) = \bar{v}_i(u) + C(t) \tag{6.10}$$

with an arbitrary function $C(t)$ and $\bar{v}_i(u)$ satisfying the first set (2.11) of conditions of theorem 2.1. Then the second set of conditions (2.12) is trivially satisfied.

We consider in detail here only case (a) which does not restrict the t dependence of the coefficients $v_i(u,t)$, but rather their dependence on u. We assume that the functions $\alpha_{ji}(t)$ for different values of the labels are not proportional to each other.

Equations (6.5) for $v_i(u,t)$ now take the form

$$\begin{aligned} v_{i,u^j}(u,t) &= \alpha_{ji}(t)\psi^i_{u^j}(u)\omega(u) \\ v_j(u,t) - v_i(u,t) &= \alpha_{ji}(t)\omega(u) \end{aligned} \qquad j \neq i. \tag{6.11}$$

The compatibility conditions $v_{i,u^j u^k} = v_{i,u^k u^j}$ for the first set of these equations have the form

$$\alpha_{ji}(t)\left(\omega(u)\psi^i_{u^j}(u)\right)_{u^k} = \alpha_{ki}(t)\left(\omega(u)\psi^i_{u^k}(u)\right)_{u^j} \qquad i \neq j \neq k. \tag{6.12}$$

These equalities imply the conditions

$$\left(\omega(u)\psi^i_{u^j}(u)\right)_{u^k} = 0 \qquad i \neq j \neq k. \tag{6.13}$$

These equations have the solution

$$\psi^i_{u^j}(u) = \frac{1}{\omega(u)}C^i_j(u^i, u^j) \qquad i \neq j. \tag{6.14}$$

The compatibility conditions of these equations $\psi^i_{u^j u^k} = \psi^i_{u^k u^j}$ lead to the relations

$$C^i_j\omega_{u^k} - C^i_k\omega_{u^j} = 0 \qquad i \neq j \neq k. \tag{6.15}$$

Using the expressions (6.14) for $\psi^i_{u^j}$ in equations (6.11) we obtain

$$\begin{aligned} v_{i,u^j}(u,t) &= \alpha_{ji}(t)C^i_j(u^i, u^j) \\ v_j(u,t) - v_i(u,t) &= \alpha_{ji}(t)\omega(u) \end{aligned} \qquad j \neq i. \tag{6.16}$$

Certain symmetry properties of α_{ij} follow from the second of these equations, namely

$$\alpha_{ij}(t) = -\alpha_{ji}(t) \qquad \alpha_{kj}(t) + \alpha_{ji}(t) = \alpha_{ki}(t). \tag{6.17}$$

Interchanging the indices i and j in the first equation of (6.16) and substituting v_j from the second equation of (6.16) to its left-hand side leads to the result

$$v_{i,u^i}(u,t) + \omega_{u^i}(u)\alpha_{ji}(t) = C_i^j(u^j, u^i)\alpha_{ij}(t). \tag{6.18}$$

To exclude the derivative v_{i,u^i}, we take the difference of this equation with the equation obtained by the substitution $j \mapsto k$ with the result

$$\omega_{u^i}(u)(\alpha_{ji}(t) - \alpha_{ki}(t)) = C_i^j(u^j, u^i)\alpha_{ij}(t) - C_i^k(u^k, u^i)\alpha_{ik}(t). \tag{6.19}$$

Since the different quantities $\alpha_{ji}(t)$ are assumed not to be proportional, this equation implies the equalities

$$C_i^j(u^j, u^i) = C_i^k(u^k, u^i) \equiv -C_i(u^i). \tag{6.20}$$

Equation (6.19) and the symmetry properties (6.17) imply the equalities

$$\omega_{u^i}(u) = C_i(u^i) \equiv a_i'(u^i) \qquad i = 1, 2, \ldots, n \tag{6.21}$$

where primes denote derivatives of functions of one variable. This gives the following expression for $\omega(u)$:

$$\omega(u) = \sum_{i=1}^{n} a_i(u^i). \tag{6.22}$$

Returning to the linear system (6.15), we easily check that expressions (6.20) and (6.21) satisfy this system identically.

Substituting these expressions for $\omega(u)$ and C_i^j into equation (6.18) we obtain

$$v_{i,u^i}(u,t) = 0 \tag{6.23}$$

which means that the system must be weakly nonlinear. In view of the formulae (6.20) and (6.21) the first of equations (6.16) takes the form

$$v_{i,u^j}(u,t) = -a_j'(u^j)\alpha_{ji}(t) \qquad j \neq i. \tag{6.24}$$

The two last equations completely determine the coefficients $v_i(u,t)$ of our system up to an arbitrary term $\gamma_i(t)$ independent of u

$$v_i(u,t) = \sum_{k \neq i} \alpha_{ik}(t) a_k(u^k) + \gamma_i(t) \tag{6.25}$$

where $\alpha_{ik}(t)$ are arbitrary functions satisfying the symmetry relations (6.17) and the functions $a_k(u^k)$ are completely arbitrary. In order to satisfy the symmetry relations identically we put

$$\alpha_{ji}(t) = \alpha_j(t) - \alpha_i(t) \tag{6.26}$$

where the functions $\alpha_i(t)$ are completely arbitrary. Substituting the expressions (6.22) and (6.26) into the second of equations (6.16) we obtain the result

$$v_j(u,t) - v_i(u,t) = \big(\alpha_j(t) - \alpha_i(t)\big) \sum_{l=1}^{n} a_l(u^l). \tag{6.27}$$

Substituting the expression (6.25) for v_i and (6.26) for α_{ji} into this equation we obtain

$$\gamma_j(t) = \gamma_i(t) \equiv \gamma(t) \tag{6.28}$$

and we end up with the following expression for v_i:

$$v_i(u,t) = \sum_{k\neq i} \big(\alpha_i(t) - \alpha_k(t)\big) a_k(u^k) + \gamma(t). \tag{6.29}$$

Thus we have obtained an example of a system of the general form (2.1) which satisfies all of the conditions of theorem 2.1

$$u^i_t = \left[\sum_{k\neq i} \big(\alpha_i(t) - \alpha_k(t)\big) a_k(u^k) + \gamma(t) \right] u^i_x \qquad i = 1, 2, \ldots, n. \tag{6.30}$$

There is still an unnecessary generality here because we can get rid of arbitrary functions $a_k(u^k)$ by a transformation of the unknowns $\tilde{u}^i = a_i(u^i)$. Dropping the tilde we obtain the final form of the example of a *t-dependent hydrodynamic-type system which admits an infinite set of hydrodynamic symmetries and which satisfies the conditions of theorem 2.1 in the case when $\beta = 0$*. The system is

$$u^i_t = \left[\sum_{k\neq i} \big(\alpha_i(t) - \alpha_k(t)\big) u^k + \gamma(t) \right] u^i_x \qquad i = 1, 2, \ldots, n. \tag{6.31}$$

It depends essentially upon n arbitrary functions of t because an overall factor depending only on t can be eliminated by suitably redefining t. Hence we can consider only two essentially different cases when $\gamma = 1$ or 0.

Let us now proceed with the actual task of this section, namely solving the system (6.31).

6.2. General solution of a hydrodynamic-type system with an infinite set of hydrodynamic symmetries

Let us first calculate the hydrodynamic symmetries of the system (6.31). We substitute the expressions (6.29) with $a_k(u^k) = u^k$ for the coefficients v_i of this system into the formula (6.4) determining the connection coefficients $\Gamma^i_{ij}(u)$ defined by equation (6.3). The result is that they do not depend on the indices i, j

$$\Gamma^i_{ij}(u) = \psi^i_{u^j}(u) = -\frac{1}{\sum_{k=1}^n u^k} \qquad i \neq j. \tag{6.32}$$

With these expressions for Γ^i_{ij} the linear equations (2.15) for the functions $w_i(u)$ take the form

$$w_{i,u^j}(u) = \frac{w_i - w_j}{\sum_{k=1}^n u^k} \qquad j \neq i. \tag{6.33}$$

For simplicity we introduce the notation $l(u) = \sum_{k=1}^n u^k$. Interchanging i and j in the last equations we obtain the system

$$w_{j,u^i}(u) = \frac{1}{l(u)} (w_j - w_i) \qquad i \neq j. \tag{6.34}$$

Excluding w_j in these equations with the aid of equations (6.33) we obtain a system of second-order equations for each w_i

$$\left[\frac{1}{l(u)} w_{i,u^i} \right]_{u^j} = 0 \qquad j \neq i \tag{6.35}$$

with the first integral

$$w_{i,u^i}(u) = l(u) b''_i(u^i) \tag{6.36}$$

where two primes denote second derivatives and $b_i(u^i)$ are arbitrary smooth functions of one variable u^i. Integrating these equations we obtain

$$w_i(u) = l(u)b_i'(u^i) - b_i(u^i) + \sum_{k\neq i} c_{ik}(u^k) \tag{6.37}$$

where $c_{ik}(u^k)$ are arbitrary smooth functions of one variable. Changing i into j we obtain the expression for w_j and substitute both expressions for the derivatives of w_i, w_j into the equations

$$w_{i,u^j} + w_{j,u^i} = 0 \qquad i \neq j \tag{6.38}$$

obtained by combining the systems (6.33) and (6.34). We perform a separation of variables u^i and u^j in the resulting equations

$$b_i'(u^i) + c_{ji}'(u^i) = -\left[b_j'(u^j) + c_{ij}'(u^j)\right] = \lambda_{ij} = \text{constant} \qquad i \neq j \tag{6.39}$$

and obtain the expressions for c_{ik} in terms of b_k, λ_{ik}

$$c_{ik}(u^k) = -b_k(u^k) - \lambda_{ik}u^k \qquad k \neq i. \tag{6.40}$$

Let us substitute these expressions into the formula (6.37) for w_i

$$w_i(u) = l(u)b_i'(u^i) - b_i(u^i) - b_j(u^j) - \lambda_{ij}u^j - \sum_{k\neq i,j} b_k(u^k) - \sum_{k\neq i,j} \lambda_{ik}u^k. \tag{6.41}$$

We interchange i and j and obtain a formula for w_j. Substituting these expressions for w_i and w_j into the original linear system (6.33) we obtain the equations

$$\lambda_{ij} \sum_{k\neq i,j} u^k = \sum_{k\neq i,j} (\lambda_{ik} - \lambda_{jk})u^k \qquad i \neq j.$$

Comparing coefficients of u^k we obtain

$$\lambda_{ik} = \lambda_{ij} + \lambda_{jk} \qquad i \neq j \neq k. \tag{6.42}$$

These symmetry conditions together with the skew symmetry property

$$\lambda_{ji} = -\lambda_{ij}$$

which follows from the definition (6.39) of λ_{ij}, are identically satisfied by putting

$$\lambda_{ij} = \mu_i - \mu_j$$

with arbitrary constants μ_i. We substitute these expressions for λ_{ij} into the formula (6.41) for w_i and express the result in terms of the new arbitrary functions of one variable

$$\beta_i(u^i) = -b_i(u^i) + \mu_i u^i. \tag{6.43}$$

We obtain the final expression for the general solution $w_i(u)$ of the linear system (2.15) which in our example has the form (6.33). The general solution of equation (6.33) is

$$w_i(u) = \sum_{k=1}^{n} \beta_k(u^k) - \beta_i'(u^i) \sum_{k=1}^{n} u^k \qquad i = 1, 2, \ldots, n. \tag{6.44}$$

We see that this solution depends on n arbitrary functions $\beta_k(u^k)$ of one variable each. This is just the right number to satisfy arbitrary Cauchy conditions for the linear system (6.33). At least in principle that also makes it possible to satisfy arbitrary Cauchy conditions for the nonlinear system (6.31). Thus in our example it was possible to integrate the linear system (2.15) completely and we did not need to use the recursions of its solutions which were derived in the preceding section. We mention that in this example the recursions do not generate new solutions (since we already know the general solution).

The hydrodynamic symmetries of the system (6.31) are generated by hydrodynamic flows of the form (2.5) with the coefficients $A_i(u, t, x)$ determined by the formula (2.14) with $C = 1$

$$u^i_\tau = \left[\sum_{k=1}^{n} \beta_k(u^k) - \beta_i'(u^i) \sum_{k=1}^{n} u^k + x + \int_0^t v_i(u, t)\, \mathrm{d}t\right] u^i_x \qquad i = 1, 2, \ldots, n \tag{6.45}$$

where we must substitute the expressions for the coefficients v_i of the system (6.31)

$$v_i(u, t) = \sum_{k\neq i} \big[\alpha_i(t) - \alpha_k(t)\big] u^k + \gamma(t). \tag{6.46}$$

Invariant solutions of the system (6.31) are determined by the condition $u^i_\tau = 0$ which for the non-singular case $u^i_x \neq 0$ implies that the expressions in square brackets in the formula (6.45) are equal to zero

$$\sum_{k=1}^{n} \beta_k(u^k) - \beta_i'(u^i) \sum_{k=1}^{n} u^k + x + \sum_{k\neq i} u^k \int_0^t \big[\alpha_i(t) - \alpha_k(t)\big]\, \mathrm{d}t + \int_0^t \gamma(t)\, \mathrm{d}t = 0$$
$$i = 1, \ldots, n. \tag{6.47}$$

In the general case in this way we obtain the linearizing transformation (3.2).

In section 3 we have already given a general proof that the equalities (3.2), determining a linearizing transformation, are compatible with the original system (2.1) if the functions $w_i(u)$ satisfy the linear system (2.15). Nevertheless, it is instructive to give an independent proof that the invariant solutions (6.47) do really exist, i.e. the equalities (6.47) are compatible with the original system (6.31) and hence they determine a general solution of the system (6.31) in an implicit form.

At first we change i into j in the equalities (6.47) and take the difference of the new one and the old one. After numerous cancellations we obtain

$$\beta_i'(u^i) - \beta_j'(u^j) = \int_0^t \big[\alpha_i(t) - \alpha_j(t)\big]\, \mathrm{d}t. \tag{6.48}$$

We note that these equalities coincide with the result of differentiating equations (6.47) with respect to u^j with $j \neq i$, which means that all non-diagonal elements of the Jacobian matrix of a system (6.47) are zero. Hence existence conditions for an implicit vector-function $u = (u^1, \ldots, u^n)$ determined by equations (6.47) mean that all diagonal elements of the Jacobian matrix, must be non-zero. We obtain these elements by differentiating the left-hand side of each of equations (6.47) with a fixed value of i with respect to u^i (with the same value of i)

$$-\beta_i''(u^i) \sum_{k=1}^{n} u^k \neq 0.$$

Thus we obtain the existence condition for an implicit function determined by equations (6.47)

$$\prod_{i=1}^{n} \beta_i''(u^i) \neq 0. \tag{6.49}$$

Next we differentiate equations (6.47) with respect to t and x with the results

$$\sum_{k=1}^{n} \beta_k'(u^k)u_t^k - \beta_i''(u^i)u_t^i \sum_{k=1}^{n} u^k - \beta_i'(u^i) \sum_{k=1}^{n} u_t^k + \sum_{k\neq i} u_t^k \int_0^t \big[\alpha_i(t) - \alpha_k(t)\big]\,\mathrm{d}t$$
$$+ \sum_{k\neq i} u^k\big[\alpha_i(t) - \alpha_k(t)\big] + \gamma(t) = 0 \tag{6.50}$$

$$\sum_{k=1}^{n} \beta_k'(u^k)u_x^k - \beta_i''(u^i)u_x^i \sum_{k=1}^{n} u^k - \beta_i'(u^i) \sum_{k=1}^{n} u_x^k + 1 + \sum_{k\neq i} u_x^k \int_0^t \big[\alpha_i(t) - \alpha_k(t)\big]\,\mathrm{d}t = 0. \tag{6.51}$$

Express the integral terms with the aid of equations (6.48) changing j to k. After simplifying we obtain the equalities

$$\beta_i''(u^i) \sum_{k=1}^{n} u^k \cdot u_t^i = \sum_{k=1}^{n} u^k\big[\alpha_i(t) - \alpha_k(t)\big] + \gamma(t) \tag{6.52}$$

$$\beta_i''(u^i) \sum_{k=1}^{n} u^k \cdot u_x^i = 1. \tag{6.53}$$

Using the existence condition (6.49) we can solve these equations with respect to u_t^i and u_x^i and substitute the results into equations (6.31). We see that they are identically satisfied which concludes the proof that the equalities (6.47) determine a general solution of this system.

Let us sum up the results of this section. Equation (6.31) is a hydrodynamic-type equation with infinitely many hydrodynamic symmetries. The linear equations (6.33) are the determining equations for the hydrodynamic symmetries. The general solution of equation (6.33) is given by equation (6.44). Equation (6.47) then gives a solution of the original system (6.31) in implicit form. To make the solutions explicit we must choose the arbitrary functions $\beta_k(u^k)$ and then solve n functional equations (6.47) for n unknowns $u^n(x,t)$.

7. Inhomogeneous diagonal hydrodynamic-type systems

The results obtained here for the diagonal t-dependent system (2.1), which is linear homogeneous in the derivatives of the unknowns, can be readily generalized to hydrodynamic-type systems with a 'diagonal inhomogeneity' of the form $f_i(u^i, t)$. Let us denote the unknowns in the inhomogeneous system by $r = \{r^i\}$, keeping the notation u for the unknowns in the homogeneous system. Such an inhomogeneous t-dependent system has the form

$$r_t^i = k_i(r,t)\,r_x^i + f_i(r^i,t) \qquad i = 1, 2, \ldots, n. \tag{7.1}$$

The following statement can be easily proved.

Assertion 7.1. *A t-dependent transformation $r^i = R_i(u^i, t)$, $i = 1, \ldots, n$, reduces the system (7.1) to the homogeneous diagonal system (2.1) iff it satisfies the ordinary differential equations*

$$\frac{\mathrm{d}R_i}{\mathrm{d}t} = f_i(R_i, t) \qquad i = 1, 2, \ldots, n. \tag{7.2}$$

Let $R_i = R_i(C_i, t)$ be a general solution and $U_i(R_i, t) = C_i$ be a general integral for the ith equation (7.2) with an arbitrary constant C_i. Then $r_i = R_i(u^i, t)$ is the transformation of the system (7.1) to the homogeneous system (2.1) and $u^i = U_i(r^i, t)$ is its inverse transformation.

These results take a more explicit form in the specific case when the inhomogeneities f_i do not depend explicitly on t or x: $f_i = f_i(r^i)$. The explicitly t-dependent system (7.1) takes the form

$$r^i_t = k_i(r, t)\, r^i_x + f_i(r^i) \qquad i = 1, 2, \ldots, n. \tag{7.3}$$

Consider the generic case $\prod_{i=1}^n f_i(r^i) \neq 0$. The differential equations (7.2) can be integrated in quadratures and we obtain an explicit form of the transformation to its characteristic variables u^i which reduce this system to the form (2.1):

$$u^i = \int \frac{\mathrm{d}r^i}{f_i(r^i)} - t \qquad i = 1, 2, \ldots, n. \tag{7.4}$$

It may happen, in particular, that the resulting homogeneous system would be explicitly independent of t and x.

It is clear now that existence conditions for an infinite set of hydrodynamic symmetries for the system (2.1), the explicit form of the commuting hydrodynamic flows and linearizing transformations, obtained in preceding sections, are readily transferred to the inhomogeneous systems (7.1).

8. Hydrodynamic-type equations with n space variables

Our next objective is to establish the relation between the symmetry group analysis of differential equations and the classical method of Riemann invariants in the multidimensional case of n space variables. We consider a very particular type of such a hydrodynamic-type system

$$u^i_t + \sum_{j=1}^n a^j(u, t) u^i_{x^j} = 0 \qquad i = 1, 2, \ldots, m \tag{8.1}$$

to make this relation more straightforward. We search for point symmetries of equation (8.1) which do not depend explicitly on $x^1, \ldots, x^n$ and which do not transform the unknowns $u = (u^1, \ldots, u^m)$. The dependent variables u are therefore included in the set of invariants. Thus, we search for symmetry generators of the particular form

$$X = \eta(u, t)\frac{\partial}{\partial t} + \sum_{j=1}^n \xi^j(u, t)\frac{\partial}{\partial x^j} \tag{8.2}$$

which are admitted by the system (8.1). Then Lie group analysis provides the following result for the symmetry generator:

$$X = \alpha(u, t) L + \sum_{j=1}^n \beta_j(u)\frac{\partial}{\partial x^j} \tag{8.3}$$

where L is the linear differential operator

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^n a^j(u, t)\frac{\partial}{\partial x^j} \tag{8.4}$$

and $\alpha(u, t)$, $\beta_j(u)$ are arbitrary functions. We note that the prolongation $\hat{L}$ of the operator L

$$\hat{L} = D_t + \sum_{j=1}^n a^j(u, t) D_{x^j} \tag{8.5}$$

where D_t and D_{x^j} denote the operators of total derivatives in t and x^j, respectively, figures in equations (8.1) so that they can be rewritten as

$$\hat{L}u^i = 0 \qquad i = 1, 2, \ldots, m. \tag{8.6}$$

Since invariant solutions must depend only on invariants we need to obtain a set of all functionally independent invariants. They are determined by the invariance condition

$$X\Phi = \alpha(u,t)\Phi_t + \sum_{j=1}^{n}\big(a^j(u,t)\alpha(u,t) + \beta_j(u)\big)\Phi_{x^j} = 0. \tag{8.7}$$

To solve it, we construct the characteristic system of ODEs

$$\frac{\mathrm{d}x^j}{\mathrm{d}t} = a^j(u,t) + \frac{\beta_j(u)}{\alpha(u,t)} \tag{8.8}$$

with the obvious solution

$$r^j = x^j - \int_{u=\text{constant}} \left[a^j(u,t) + \frac{\beta_j(u)}{\alpha(u,t)}\right] \mathrm{d}t = c_j \tag{8.9}$$

with constant c_j. So, including the set of obvious invariants u, we obtain the general form of the invariant satisfying equation (8.7) as

$$\Phi = \Phi(r^1, r^2, \ldots, r^n, u^1, u^2, \ldots, u^m) \tag{8.10}$$

where Φ is an arbitrary smooth function and its arguments form the basis, i.e. a complete set of functionally independent invariants.

The general form of an invariant solution will hence be

$$u^i = f^i(r^1, r^2, \ldots, r^n) \qquad i = 1, 2, \ldots, m \tag{8.11}$$

where both u^i and r^j are invariant under the point group generated by the vector field (8.3). Following the standard method of symmetry reduction, we substitute the reduction formula (8.11) into the original equations (8.1). We obtain the reduced system of differential equations

$$\sum_{j=1}^{n} \beta_j(u)\frac{\partial f^i}{\partial r^j}(r^1, \ldots, r^n) = 0 \qquad i = 1, \ldots, m. \tag{8.12}$$

The functions $\beta_j(u)$ are really functions of the invariants r^i, since we must use equation (8.11) to express u as an argument of β_j in terms of r^i. Thus equation (8.12) is a system of m differential equations for m functions $f^i(r^1, \ldots, r^n)$. These equations are still nonlinear, like the original equations (8.1). The reduction is in the fact that (8.1) involves $n + 1$ independent variables, namely $(t, x^1, \ldots, x^n)$, whereas equation (8.12) involves only n independent variables, namely $(r^1, \ldots, r^n)$. Once equation (8.12) is solved and the functions f^j are known, then equation (8.11) determines the functions $u^i(t, x^1, \ldots, x^n)$ implicitly.

Let us now discuss the degree of freedom in the obtained solutions. The variables r^j are Riemann invariants [15] and the functions $\beta_j(u)$ and $\alpha(u,t)$ in them are *a priori* arbitrary. However, equation (8.12) is a first-order differential constraint on the functions $f^i(\vec{r})$. The extent to which equation (8.12) constrains these functions depends on the dimension of the space spanned by the vectors

$$\gamma_1 = f_{,r^1} \quad \ldots \quad \gamma_n = f_{,r^n} \qquad \gamma_i \in R^m \quad i = 1, \ldots, n. \tag{8.13}$$

In particular, if all of these vectors are linearly independent, then we must have $\beta_j(u) = 0$, $j = 1, \ldots, n$. The expression for the Riemann invariants then simplifies to

$$r^j = x^j - \int_{u=\text{constant}} a^j(u, t)\,\mathrm{d}t \qquad j = 1, \ldots, n. \tag{8.14}$$

The variables r^j in this case are completely specified, but the functions $f^i(\vec{r})$ are completely arbitrary. Vice versa, if the set of vectors (8.13) is not linearly independent, then some of the functions $\beta_j(u)$ will be free. These functions will figure in the variables r^j, via equation (8.9). The functions f^i are then constrained by equation (8.12).

Let us now discuss the connection between the symmetry reduction method described above and the generalized method of characteristics [35] for a homogeneous first-order quasilinear system:

$$\sum_{\mu=0}^{n} A^\mu(u, x^0)\frac{\partial u}{\partial x^\mu} = 0 \tag{8.15}$$

where A^μ are, in general, some $m \times m$ matrix functions. In our example (8.1) they are all proportional to the identity matrix I:

$$A^0 = \mathrm{I} \qquad A^1 = a^1(u, x^0)\mathrm{I} \quad \cdots \quad A^n = a^n(u, x^0)\mathrm{I}. \tag{8.16}$$

Here we denote the independent variables by $x = (x^0 = t, x^1, \ldots, x^n)$.

The methodological approach assumed in the generalized method of characteristics [30] is an algebraization of the partial differential equations (8.15). We assume that there exists a set of real-valued functions $\xi, \lambda_0, \ldots, \lambda_n, \gamma^1, \ldots, \gamma^m$, such that all derivatives of u^i with respect to x^μ are decomposable as follows:

$$\frac{\partial u^i}{\partial x^\mu} = \xi\lambda_\mu\gamma^i \qquad i = 1, \ldots, m \quad \mu = 0, \ldots, n. \tag{8.17}$$

The function $\xi(x)$ depends on x alone, λ_μ and γ^i depend on x^0 directly and also through the dependent variables u. Substituting (8.17) into equation (8.15) we obtain

$$\sum_{\mu=0}^{n}\sum_{i=1}^{m}\big(\lambda_\mu (A^\mu)^s_i\big)\gamma^i = 0 \qquad s = 1, \ldots, m. \tag{8.18}$$

The necessary and sufficient condition for the existence of a non-trivial solution $\gamma = (\gamma^1, \ldots, \gamma^m)$ of equation (8.18) is

$$\det\left(\sum_{\mu=0}^{n}\lambda_\mu (A^\mu)^s_i\right) = 0 \iff \sum_{\mu=0}^{n}\lambda_\mu a^\mu = 0 \quad a^0 = 1. \tag{8.19}$$

First, we must find the 'wavevectors' $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ such that (8.19) holds. For each such vector we find the 'polarization vector' γ satisfying equation (8.18).

In our case the linearly independent wavevectors λ^l admitted by equations (8.15) have the form

$$\lambda^1 = \big(-a^1(u, x^0), 1, 0, \ldots, 0\big) \quad \cdots \quad \lambda^n = \big(-a^n(u, x^0), 0, \ldots, 0, 1\big) \tag{8.20}$$

and satisfy the conditions

$$\sum_{\mu=0}^{n}\lambda^l_\mu a^\mu = 0 \qquad l = 1, \ldots, n. \tag{8.21}$$

Equations (8.21) are n independent relations between the coefficients $a^0, \ldots, a^n$. Relations (8.21) imply that the vectors γ_l are arbitrary, since equations (8.18) are solved identically. The construction of a Riemann n-wave requires that the distribution of linearly independent vector fields $\{\gamma_1, \ldots, \gamma_n\}$ be Abelian (in the Frobenius sense) [27, 30, 36]. This is satisfied in our case.

We obtain a surface S given by equation (8.11) parametrized by n parameters $r^1, \ldots, r^n$. It satisfies the system

$$\frac{\partial f^i}{\partial r^j} = \gamma^i_j(f^1, \ldots, f^m) \tag{8.22}$$

however, since the vectors γ_l are arbitrary, the functions f^i are also arbitrary.

We now take the differential $\mathrm{d}f$ of equation (8.11) and use equation (8.17), use the linear independence of the vectors $(\gamma_1, \ldots, \gamma_n)$ and obtain

$$\mathrm{d}r^j = \sum_{\mu=0}^{n} \xi(x)\lambda^j_\mu(r^1, \ldots, r^n, x^0)\,\mathrm{d}x^\mu. \tag{8.23}$$

The general integral of this Pfaffian system coincides with the group invariants (8.14), once equations (8.20) are taken into account.

Thus the relation between symmetry reduction and the generalized method of characteristics for equation (8.1) is established.

9. Conclusions

The hydrodynamic flows used in section 2 are generalized symmetries, in the sense that they are neither point, nor contact ones. However, they share an important property of these symmetries: they depend only on first derivatives of the dependent variables, moreover the dependence is linear. They turn out to be very useful for solving hydrodynamic-type equations. Indeed, we have shown that if a system allows an infinite set of hydrodynamic symmetries, it is linearizable. The linearization provides us with very general solutions of the original nonlinear system. Indeed, the solution of equation (2.1) obtained in this manner will depend on n arbitrary functions of one variable. This is just the right amount of generality needed to satisfy arbitrary Cauchy conditions (though this does not exclude the existence of other solutions, not obtainable in this manner).

The linearization via hydrodynamic symmetries is not equivalent to the existence of a Lax pair, nor is it a 'direct linearization', meaning an invertible transformation of variables, taking a nonlinear equation into a linear one.

What is happening is that the (linear) determining equations for the symmetries have a very general set of solutions. It is so general that the method of symmetry reduction leading to invariant solutions, provides virtually all solutions of the system.

This linearization was obtained by Tsarev [3, 4] for hydrodynamic-type equations with time- and coordinate-independent coefficients. We have generalized the result to include the dependence on x and t. Moreover, we have provided an explicit group-theoretical basis for the linearization, opening the way for further generalizations and applications.

The conditions of theorem 2.1, necessary and sufficient for the existence of infinitely many symmetries, are quite restrictive. This is not surprising since we know that 'interesting' linearizable nonlinear equations are rare. The example of section 6 shows that the set of such equations is not empty. Indeed, we have provided a family of such equations depending on n functions of time (see equation (6.31)).

Finally, we mention that the set of hydrodynamic symmetries may be smaller, i.e. finite, or depending on fewer arbitrary functions. Even then these symmetries can be used to obtain special solutions via symmetry reduction, or via a simultaneous solution of the original equations and the additional commuting flows.

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